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On Integral Invariants.

By F. W. Reed.

The work of Poincaré* on integral invariants has been extended along the same lines by De Donder† and Goursat.‡ Lie§ has written upon a class of linear integral invariants somewhat more general than those considered by Poincaré.

In the present paper the method of infinitesimal transformations developed by Lie‖ is applied to those types defined and discussed by Poincaré. The conditions for invariants of order p are found as a system of ordinary linear differential equations, which are the same as those found by Goursat. The relation between the solutions of the systems of various orders is brought out in full. In particular it is shown how all the linear invariants depend upon the original system and how all the invariants of higher order can be constructed when these are known. An extension of Poisson's theorem is given. Further it is shown that the least action integral of dynamics is nothing other than an integral invariant of the equations of motion.

§1. *Invariants of Order p.*

Consider the multiple integral

$$I_p=\int^p\Sigma A_{\alpha_1\ldots\ldots\alpha_p}dx_{\alpha_1}dx_{\alpha_2}\ldots\ldots dx_{\alpha_p}, \tag{a}$$

where

$$A_{\alpha_1\ldots\ldots\alpha_p}=\pm A_{\alpha'_1\ldots\ldots\alpha'_p},$$

the upper or lower sign being taken according as the rearrangement of the $\alpha_1\ldots\ldots\alpha_p$ in the new order $\alpha'_1\ldots\ldots\alpha'_p$ is obtained by an even or odd number of consecutive changes, and the Σ sign is extended to all the combinations of the

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*Poincaré, *Journal de l'École Polytechnique,* 2^e^ série, premier cahier (1895). "Les Méthodes Nouvelles de la Mécanique Céleste, Tome III; *Acta Mathematica,* 13 (1890).

† De Donder, *Rendiconti del Circolo Matematico di Palermo,* 15, 16 (1901, 1902).

‡ Goursat, *Journal de Mathématiques pures et appliquées,* 6e série, Tome IV (1908).

§ Lie, "Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig," 49 (1897), pp. 342, 369.

‖ Lie, "Theorie der Transformationsgruppen."

n letters $\alpha_1 \ldots . \alpha_n$ taken p at a time. This integral is equivalent to the integral

$$I_p = \int^p \Sigma A_{\alpha_1 \ldots . \alpha_p} \frac{\partial(x_{\alpha_1} \ldots . x_{\alpha_p})}{\partial(y_1 \ldots . y_p)} dy_1 \ldots . dy_p . \qquad \text{(b)}$$

The interchange of x_{α_i} and $x_{\alpha_{i+1}}$ in any one of the functional determinants changes the sign of the corresponding term in (b), but the sign may be restored by making the same permutation of α_i and α_{i+1} in the subscript of $A_{\alpha_1 \ldots . \alpha_p}$. The same changes have the same effect upon any particular term of (a). It is sufficient, therefore, when a choice of the subscripts of $A_{\alpha_1 \ldots . \alpha_p}$ is once made to write the subscripts in $dx_{\alpha_1} \ldots . dx_{\alpha_p}$ in the *same order.*

To facilitate the writing the following symbols are introduced:

$$(1 \ldots . p-2, \lambda, p) \equiv A_{\alpha_1 \ldots . \alpha_{p-2}, \lambda, \alpha_p}, \text{ etc.}$$

$$[1, \lambda] = \frac{\partial X_{\alpha_1}}{\partial x_\lambda}, \text{etc.} \quad [\lambda, p] = \frac{\partial X_\lambda}{\partial x_{\alpha_p}}.$$

Under the transformation

$$dx_i = X_i dt, \qquad (1)$$

or

$$X(f) = \frac{\partial f}{\partial x_1} X_1 + \frac{\partial f}{\partial x_2} X_2 + \ldots . + \frac{\partial f}{\partial x_n} X_n,$$

the integral (a) becomes

$$I'_p = \int^p \sum_{\alpha_1 \ldots . \alpha_p} [(1 \ldots . p) + X(1 \ldots . p) dt] [dx_{\alpha_1} + \Sigma [1, \lambda] dx_\lambda dt]$$
$$[dx_{\alpha_2} + \Sigma [2, \lambda] dx_\lambda dt] \ldots . [dx_{\alpha_p} + \Sigma [p, \lambda] dx_\lambda dt]. \qquad \text{(a')}$$

The term of zero degree in dt is I_p of (a). Equating to zero the coefficient of $dx_{\alpha_1} \ldots . dx_{\alpha_p}$ in the term of first degree in dt we have

$$\Sigma [X(1 \ldots . p) + (1 \ldots . p-1, \lambda)[\lambda, p] + (1 \ldots . p-2, \lambda, p)[\lambda, p-1]$$
$$+ \ldots . + (\lambda, 2 \ldots . p)[\lambda, 1]] = 0. \qquad (2)$$

If these conditions are satisfied I_p is an integral invariant under the transformation.

THEOREM I. *If an I_1 and an I_p are known, an I_{p+1} may be written by means of the relation*

$$(1 \ldots . p+1) = (1)(2 \ldots . p+1) - (2)(1, 3 \ldots . p+1) + \ldots . \pm (p+1)(1 \ldots . p).$$

Substitute this expression for $(1 \ldots . p+1)$ in the general condition (2). The respective terms become

$$X(1 \ldots . p+1) = X((1)(2 \ldots . p+1) - (2)(1, 3 \ldots . p+1)$$
$$+ \ldots . \pm (p+1)(1 \ldots . p)),$$

$$\begin{aligned}
\Sigma(1\ldots.p,\lambda)[\lambda,p+1] &= \Sigma[(1)(2\ldots.p,\lambda)-(2)(1,3\ldots.p,\lambda) \\
&\quad +\ldots.\pm(\lambda)(1\ldots.p)][\lambda,p+1], \qquad \lambda(=)\alpha_{p+1}, \\
\Sigma(1\ldots.p-1,\lambda,p+1)[\lambda,p] &= \Sigma[(1)(2\ldots.p-1,\lambda,p+1) \\
&\quad -(2)(1,3\ldots.p-1,\lambda,p+1) \\
&\quad +\ldots.\pm(p+1)(1\ldots.p-1,\lambda)][\lambda,p], \quad \lambda(=)\alpha_p,
\end{aligned}$$

……

$$\begin{aligned}
\Sigma(1,\lambda,3\ldots.p+1)[\lambda,2] &= \Sigma[(1)(\lambda,3\ldots.p+1)-(\lambda)(1,3\ldots.p+1) \\
&\quad +\ldots.\pm(p+1)(1,\lambda,3\ldots.p)][\lambda,2], \quad \lambda(=)\alpha_2, \\
\Sigma(\lambda,2\ldots.p+1)[\lambda,1] &= \Sigma[(\lambda)(2\ldots.p+1)-(2)(\lambda,3\ldots.p+1) \\
&\quad +\ldots.\pm(p+1)(\lambda,2\ldots.p)][\lambda,1], \qquad \lambda(=)\alpha_1,
\end{aligned}$$

where $\lambda(=)\alpha_i$ means λ different for all the $\alpha_1\ldots.\alpha_{p+1}$ except α_i. By adding to the second equation

$$\begin{aligned}
0= &(1)(2\ldots.p,1)[1,p+1]-(2)(1,3\ldots.p,2)[2,p+1] \\
&+\ldots.\pm(p)(1\ldots.p-1,p)[p,p+1] \\
&\pm[(1)(1\ldots.p)[1,p+1]+(2)(1\ldots.p)[2,p+1] \\
&+\ldots.+(p)(1\ldots.p)[p,p+1]],
\end{aligned}$$

the gaps are filled and we have

$$\begin{aligned}
\Sigma(1\ldots.p,\lambda)[\lambda,p+1] &= \Sigma[(1)(2\ldots.p,\lambda)-(2)(1,3\ldots.p,\lambda) \\
&\quad +\ldots.\pm(\lambda)(1\ldots.p)[\lambda,p+1]], \quad \lambda(=)\alpha_{p+1},
\end{aligned}$$

where λ takes all values consistent with the properties of $(1\ldots.p)$ previously indicated. The remaining equations may be reduced in a similar manner. Adding now the first terms in the right members of the equations thus reduced the sum is

$$\begin{aligned}
(1)\Sigma[X(2\ldots.p+1)+(2\ldots.p,\lambda)[\lambda,p+1]+\ldots.+(\lambda,3\ldots.p+1)[\lambda,2]] \\
+(2\ldots.p+1)\Sigma[X(1)+(\lambda)[\lambda,1]].
\end{aligned}$$

This expression is zero when the (i) and the $(1\ldots.p)$ are coefficients in an I_1 and an I_p, respectively. The sum of the second terms vanish in a similar manner, etc. Thus the conditions for an I_{p+1} are fulfilled.

Theorem II. *The product of p linearly independent invariants I_1 is an I_p.*

Suppose p linear invariants are known

$$I_1^{(i)}=\int L_i \qquad (i=1,2\ldots.p),$$

where

$$\begin{aligned}
L_1 &= A_1^1dx_1+\ldots.+A_n^1dx_n=(1)^1dx_1+\ldots.+(n)^1dx_n, \\
L_2 &= A_1^2dx_1+\ldots.+A_n^2dx_n=(1)^2dx_1+\ldots.+(n)^2dx_n, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
L_p &= A_1^pdx_1+\ldots.+A_n^pdx_n=(1)^pdx_1+\ldots.+(n)^pdx_n.
\end{aligned}$$

The determinants of the matrix of the right-hand members of the system give all the terms of this product since products in the same row could not enter, and those from the same column give zero on integration. Further, the coefficients of $dx_{\alpha_1}\ldots dx_{\alpha_p}$ is $\Sigma\pm(1)^1(2)^2\ldots(p)^p$, where the summation means all the permutations of $\alpha_1\ldots\alpha_p$, and the upper or lower sign is to be taken according as it requires an even or an odd number of changes to bring an element of this sum to the order $\alpha_1\ldots\alpha_p$. Consequently, remembering the properties of $(1\ldots p)$ previously defined, we have

$$\Sigma\pm(1)^1(2)^2\ldots(p)^p=(1\ldots p).$$

COROLLARY 1. *If the L_i are not linearly independent the resulting I_p is identically zero.*

COROLLARY 2. *Since the determinants may be expanded with respect to their minors of any order we have*

$$(1\ldots p)(p+1\ldots p+q)=(1\ldots p+q)$$

which gives symbolically $I_{p+q}=I_pI_q$.

COROLLARY 3. *When p is odd $(I_p)^2\equiv 0$.*

COROLLARY 4. *The number of coefficients in an I_p is $\frac{n!}{p!(n-p)!}$, the same as the number of terms in the product $L_1L_2\ldots L_p$, consequently, when n independent linear invariants are known all the I_p can be written.*

The converse of this last corollary is not true. For example, the invariant of the second order $I_1^1I_1^2+I_1^3I_1^4$ is not factorable.

THEOREM III. *Knowing an I_{p+1}, an I_p can be found by means of the formula*

$$(1\ldots p)=\Sigma(1\ldots p,\sigma)X_\sigma.$$

To verify this formula substitute it in equation (2). We get

$$\begin{array}{ll}\Sigma[X(1\ldots p,\sigma)\cdot X_\sigma+(1\ldots p,\sigma)X(X_\sigma), & \sigma\neq\alpha_1\ldots\alpha_p,\\ \quad+\sum\limits_{\lambda}[\lambda,p]\cdot\sum\limits_{\sigma}(1\ldots p-1,\lambda,\sigma)X_\sigma, & \sigma\neq\alpha_1\ldots\alpha_{p-1},\lambda,\\ \quad+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, & \\ \quad+\sum\limits_{\lambda}[\lambda,1]\cdot\sum\limits_{\sigma}(\lambda,2\ldots p,\sigma)X_\sigma=0], & \sigma\neq\lambda,\alpha_2\ldots\alpha_p.\end{array}$$

The subscripts given by $(1, 2, \ldots)$ must be chosen so that there are no repetitions. The coefficient of X_ρ, where $\rho\neq\alpha_1\ldots\alpha_p$ is

$$X(1\ldots p,\rho)+\sum_{\sigma}(1\ldots p,\sigma)[\sigma,\rho]$$
$$+\sum_{\lambda}(1\ldots p-1,\lambda,\rho)[\lambda,p]+\ldots+\sum_{\lambda}(\lambda,2\ldots p,\rho)[\lambda,1]=0.$$

These are the conditions for an I_{p+1}. The remaining terms are zero, for we have as coefficients of X_{α_i}, $(i=1, 2\ldots.p)$

$$\sum_{\sigma}(1\ldots.p, \sigma)[\sigma, i]+\sum_{\lambda}(1\ldots.i-1, \lambda, i+1\ldots.p, i)[\lambda, i].$$

Here the coefficient of any $[\sigma, i]$ is identically zero since interchanging λ and α_i is accomplished by $j+(j-1)$ moves. This number is odd.

The application of this theorem to the problem of finding invariants of order $p-1$ from those of order $p+1$ leads to coefficients identically zero. We have

$$(1\ldots.p)=\Sigma(1\ldots.p, \lambda)X_\lambda, \qquad \lambda \neq \alpha_1\ldots.\alpha_p,$$

also

$$(1\ldots.p-1)=\Sigma(1\ldots.p-1, \sigma)X_\sigma, \qquad \sigma \neq \alpha_1\ldots.\alpha_{p-1}.$$

Substituting the first equation in the second it becomes

$$(1\ldots.p-1)=\sum_{\sigma}X_\sigma\sum_{\lambda}(1\ldots.p-1, \sigma, \lambda)X_\lambda, \qquad \lambda \neq \sigma.$$

The coefficient of $X_\sigma X_\lambda$ is

$$(1\ldots.p-1, \sigma, \lambda)+(1\ldots.p-1, \lambda, \sigma)\equiv 0.$$

If the equations are canonical then

$$I_2=\int dx_1dx_2+dx_3dx_4+\ldots.$$

as can be verified readily. Applying Theorem III, we have

$$A_1=X_2, \quad A_2=-X_1, \ \ldots.,$$

$$I_1=\int -\frac{\partial F}{\partial x_1}dx_1-\frac{\partial F}{\partial x_2}dx_2-\ldots.-\frac{\partial F}{\partial x_n}dx_n.$$

The condition for the invariant of the *n-th* order is

$$X(1\ldots.n)+(1\ldots.n)\left[\frac{\partial X_1}{\partial x_1}+\ldots.+\frac{\partial X_n}{\partial x_n}\right]=0,$$

which becomes

$$\Sigma\frac{\partial}{\partial x_i}\{(1\ldots.n)X_i\}=0,$$

showing that $(1\ldots.n)$ is a multiplier of (1). In particular, if the equations are canonical,

$$\Sigma\frac{\partial X_i}{\partial x_i}=0, \text{ and } (1\ldots.n)=\text{const.}=1$$

is a multiplier.

The invariant of order $n-1$ derived from this is identically zero.

§ 2. *Invariants of the Type* $\int\sqrt[p]{\Sigma(1\ldots.p)dx_{\alpha_1}\ldots.dx_{\alpha_p}}$.

Let

$$I_p=\int\sqrt[p]{\sum_{\alpha_1\ldots\ldots\alpha_p}(i\ldots.p)dx_{\alpha_1}\ldots.dx_{\alpha_p}}=\int\sqrt[p]{A}.$$

We have

$$X(A^{\frac{1}{p}})=\frac{1}{p}\cdot A^{\frac{1}{p}-1}X(A).$$

Then I_p will be an integral invariant when

$$X(A)=0.$$

This condition, when expanded, is identical with the condition (2). However, in the present case, $(1\ldots.p)$ has the same value for all permutations of $\alpha_1\ldots.\alpha_p$, and may be different from zero when $\alpha_i=\alpha_j$ in the subscript. The conditions to be satisfied are therefore

$$\Sigma[X(1\ldots.p)+(1\ldots.p-1,\lambda)[\lambda,p]+\ldots.+(\lambda,2\ldots.p)[\lambda,1]]=0,$$

where λ takes all values 1 to n.

THEOREM IV. *An* I_{p+q} *can be constructed from an* I_p *and an* I_q *by means of the formula*

$$(1\ldots.p+q)=(1\ldots.p)(p+1\ldots.p+q).$$

Substituting this expression in the conditions for an I_{p+q}, namely,

$$\begin{aligned}X(1\ldots.p+q)+(1\ldots.p+q-1,\lambda)[\lambda,p+q]&\\+\ldots.+(\lambda,2\ldots.p+q)[\lambda,1]&=0,\end{aligned}$$

it becomes

$$\begin{aligned}(1\ldots.p)X(p+1\ldots.p+q)&+(p+1\ldots.p+q)X(1\ldots.p)\\&+(1\ldots.p)(p+1\ldots.p+q-1,\lambda)[\lambda,p+q]\\&+\ldots.+(1\ldots.p)(\lambda,p+2\ldots.p+q)[\lambda,p+1]\\&+(1\ldots.p-1,\lambda)(p+1\ldots.p+q)[\lambda,p]\\&+\ldots.+(\lambda,2\ldots.p)(p+1\ldots.p+q)[\lambda,1]=0,\end{aligned}$$

and this condition is satisfied since

$$X(1\ldots.p)+(1\ldots.p-1,\lambda)[\lambda,p]+\ldots.+(\lambda,2\ldots.p)[\lambda,1]=0$$

and

$$\begin{aligned}X(p+1\ldots.p+q)+(p+1\ldots.p+q-1,\lambda)[\lambda,p+q]&\\+\ldots.+(\lambda,p+2\ldots.p+q)[\lambda,p+1]&=0.\end{aligned}$$

COROLLARY. *If*

$$I_1^1=\int L_1,\quad I_1^2=\int L_2,\ \ldots.,\quad I_1^p=\int L_p$$

are p linear integral invariants of (1) *then*

$$\int\sqrt[p]{L_1L_2\ldots.L_p}$$

is an integral invariant.

§ 3. *Relative Invariants.*

The following definitions and theorems are due to Poincaré:

(a). An I_p extended to a closed multiplicity E_p in a space of n dimensions can be replaced by an I_{p+1} extended to a multiplicity E_{p+1} in a space of n dimensions limited by the E_p. The coefficients of the I_{p+1} are found from those of the I_p by means of the relations

$$(1\ldots.p+1)=\frac{\partial(1\ldots.p)}{\partial x_{a_{p+1}}}\pm\frac{\partial(2\ldots.p,p+1)}{\partial x_{a_1}}+\ldots.\pm\frac{\partial(p+1,1\ldots.p-1)}{\partial x_{a_p}},$$

the upper or lower signs being taken according as p is even or odd.

(b). The expression under the integral sign of I_p is an exact differential when the $(1\ldots.p+1)$ are zero.

(c). The I_{p+1} of (a) is an integral of an exact total differential. This is the generalized Stokes' theorem.

Let $A'_{a_1\ldots.a_p}\equiv(1\ldots.p)'$ represent the left-hand member of equation (2). The increment of I_p was found to be

$$[\textstyle\int^p \sum\limits_{a_1\ldots.a_p}(1\ldots.p)'dx_{a_1}\ldots.dx_{a_p}]\,dt.$$

If

$$\Sigma(1\ldots.p)'dx_{a_1}\ldots.dx_{a_p}$$

is exact, and if its integral be extended to a closed E_p, the resulting equivalent I_{p+1} is identically zero. In this case the original I_p is a relative integral invariant of (1). We shall write it J_p. When

$$(1\ldots.p)'=0, \tag{2}$$

we have an absolute invariant I_p; when

$$\Sigma(1\ldots.p)'dx_{a_1}\ldots.dx_{a_p} \textit{ is exact,} \tag{6}$$

we have a relative invariant J_p. By means of the equation (5) the statement (6) may be written in full as an equation,

$$\frac{\partial(1\ldots.p)'}{\partial x_{a_{p+1}}}\pm\frac{\partial(2\ldots.p,\,p+1)}{\partial x_{a_1}}+\ldots.\pm\frac{\partial(p+1,1\ldots.p-1)}{\partial x_{a_p}}=0. \tag{7}$$

The relation between these two types of invariants may be illustrated by some simple examples.

EXAMPLE 1. Let equations (1) have the canonical form

$$\frac{dx_1}{dt}=\frac{\partial F}{\partial x_2},\quad \frac{dx_2}{dt}=-\frac{\partial F}{\partial x_1},\quad \frac{dx_3}{dt}=\frac{\partial F}{\partial x_4},\quad \frac{dx_4}{dt}=-\frac{\partial F}{\partial x_3}.$$

We have

$$A'_i=\Sigma\left[X(A_i)+A_k\frac{\partial X_k}{\partial x_i}\right]\qquad (i,\,k=1,\,2,\,3,\,4).$$

The integral

$$\int x_2 dx_1 + x_4 dx_3$$

is a relative invariant, since

$$\frac{\partial A'_i}{\partial x_k} - \frac{\partial A'_k}{\partial x_i} = 0.$$

It is not, however, an I_1, since $A'_i \neq 0$.

EXAMPLE 2. Let equations (1) have the canonical form

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i} = X_i, \quad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i} = Y_i \qquad (i=1, 2, \ldots, n).$$

Where F is homogeneous of degree 1 in y_i, or

$$F = \Sigma y_i \frac{\partial F}{\partial y_i} = \Sigma y_i X_i.$$

The partial derivatives of this equation with respect to x_i, y_i are

$$Y_i + \sum_k y_k \frac{\partial X_k}{\partial x_i} = 0, \quad \sum_k y_k \frac{\partial X_i}{\partial y_k} = 0,$$

and these are the necessary and sufficient conditions that

$$I = \int \Sigma y_i dx_i$$

is an absolute invariant.

According to Goursat we designate as D the operation of passing from an I_p to an I_{p+1} by means of equation (5), and as E the operation of passing from an I_p to an I_{p-1} by means of equation (4).

THEOREM V. *The operations D and E are in general not inverse.*

Consider the case of a linear invariant, $I_1 = \int a x_1 + b x_2$ of the linear system of equations in x_1, x_2. By identifying (5) with (4) we have

$$a = \left(\frac{\partial a}{\partial x_2} - \frac{\partial b}{\partial x_1}\right) X_2, \quad b = -\left(\frac{\partial a}{\partial x_2} - \frac{\partial b}{\partial x_1}\right) X_1, \tag{8}$$

where a, b must satisfy the conditions (2); but (2) are satisfied only by $a = \frac{\partial F}{\partial x_1}$, $b = \frac{\partial F}{\partial x_2}$ where F is an integral of the system. With this substitution, however, the second members of (8) become identically zero, consequently, an I_1 satisfying the conditions (8) and (2) does not exist.

§ 4. *Extension of Poisson's Theorem.*

If equations (1) are in canonical form

$$\frac{dx_i}{dt}=\frac{\partial F}{\partial y_i},\quad \frac{dy_i}{dt}=-\frac{\partial F}{\partial x_i}\qquad (i=1,\,2,\,\ldots.,\,n), \tag{1}$$

where F does not contain t explicitly, then $F=$ const. is an integral of (1). Further, if we put

$$[F_1,\,F_2]\equiv\Sigma\left(\frac{\partial F_1}{\partial x_i}\,\frac{\partial F_2}{\partial y_i}-\frac{\partial F_1}{\partial y_i}\,\frac{\partial F_2}{\partial x_i}\right),$$

we have

$$\frac{dF_1}{dt}=[F_1,\,F]$$

by means of (1). The necessary and sufficient condition that $F_1=$ const. be an integral of (1) is that $[F_1,\,F]=0$. Let

$$I_1=\int\Sigma\,(M_i dx_i+N_i dy_i)$$

be a linear integral invariant of (1), then the conditions to be satisfied are

$$\frac{dM_k}{dt}+\sum_i\left(M_i\frac{\partial^2 F}{\partial y_i\partial x_k}-N_i\frac{\partial^2 F}{\partial x_i\partial x_k}\right)=0, \tag{m}$$

$$\frac{dN_k}{dt}+\sum_i\left(M_i\frac{\partial^2 F}{\partial y_i\partial y_k}-N_i\frac{\partial^2 F}{\partial x_i\partial y_k}\right)=0. \tag{n}$$

The coefficients of another similar invariant may be represented by M'_i, N'_i determined by the same conditions and lettered (m′) (n′). These $4n$ equations when multiplied by

$$N'_1,\,\ldots.N'_n,\;-M'_1,\,\ldots.-M'_n,\;-N_1,\,\ldots.-N_n,\;M_1,\,\ldots.M_n,$$

respectively, give on adding,

$$\frac{d}{dt}\,\Sigma\,(M_iN'_i-N_iM'_i)=0.$$

Therefore,

$$\Sigma\,(M_iN'_i-N_iM'_i)=\text{const.}$$

Remembering that

$$M_i=\frac{\partial F_1}{\partial x_i},\quad M'_i=\frac{\partial F_2}{\partial x_i},$$

$$N_i=-\frac{\partial F_1}{\partial y_i},\quad N'_2=-\frac{\partial F_2}{\partial y_i},$$

we have

$$\Sigma\left(\frac{\partial F_1}{\partial x_i}\,\frac{\partial F_2}{\partial y_i}-\frac{\partial F_1}{\partial y_i}\,\frac{\partial F_2}{\partial x_i}\right)\equiv[F_1,\,F_2]=\text{const}$$

It is well known (theorem of Poisson) that $[F_1, F_2]$ is an integral of (1) when F_1, F_2 are integrals. The above result can be expressed in determinant form

$$\Sigma \begin{vmatrix} M_i & M'_i \\ N_i & N'_i \end{vmatrix} = \text{const.}$$

Consider the case of four known integrals of (1). Then we have

$$\begin{vmatrix} M^1_i & M^2_i & M^3_i & M^4_i \\ N^1_i & N^2_i & N^3_i & N^4_i \\ M^1_k & M^2_k & M^3_k & M^4_k \\ N^1_k & N^2_k & N^3_k & N^4_k \end{vmatrix} = \text{const.}$$

since all the minors of the second order found in the first and second rows, also in the third and fourth rows, are constant.

And generally

$$\sum_{\alpha_1}\sum_{\alpha_2}\cdots\sum_{\alpha_p} \begin{vmatrix} M^1_{\alpha_1} & M^2_{\alpha_1}\ldots\ldots & M^{2p}_{\alpha_1} \\ N^1_{\alpha_1} & N^2_{\alpha_1}\ldots\ldots & N^{2p}_{\alpha_1} \\ \vdots & & \\ M^1_{\alpha_p} & M^2_{\alpha_p}\ldots\ldots & M^{2p}_{\alpha_p} \\ N^1_{\alpha_p} & N^2_{\alpha_p}\ldots\ldots & N^{2p}_{\alpha_p} \end{vmatrix} \equiv \sum_{\alpha_1\ldots\ldots\alpha_p} \Delta_{\alpha_1\ldots\ldots\alpha_p} = \text{const.}$$

We may assume all the α_i different in making the summation, because otherwise the particular determinant involved would be zero.

In particular if $p=n$ we have, since the total number of determinants in the sum is $n!$, and, since the permutation of the numbers $\alpha_1\ldots\ldots\alpha_p$ corresponds to a transposition of the rows keeping rows $2k-1, 2k$ together,

$$\sum_{\alpha_1\ldots\ldots\alpha_n} \Delta_{\alpha_1\ldots\ldots\alpha_n} = n!\,\Delta_{\alpha_1\ldots\ldots\alpha_n} = n!\,\Delta_{1,2\ldots\ldots n}\,.$$

$\Delta_{1\ldots\ldots n}$ is simply the functional determinant of the F_i with respect to the x_i, y_i. This is the extension of Poisson's theorem.

§ 5. *Least Action*

Consider the canonical equations of dynamics

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i},$$

which admit the relative invariant

$$J_1 = \int \Sigma p_i dq_i,$$

and the equivalent absolute invariant (by Stokes' theorem)

$$I_2 = \int\int \Sigma dp_i dq_i.$$

Goursat[*] has shown that the integral invariant of *p-th* order

$$I_p=\int_{E_p}\Sigma A_{\alpha_1\dots\alpha_p}dx_{\alpha_1}\dots dx_{\alpha_p}$$

of the system

$$\frac{dx_i}{dt}=X_i \qquad (i=1,\,2,\,\dots,\,n)$$

is identical with

$$I_p=\int_0^T dt\int_{E_{p-1}}\Sigma C_{\alpha_1\dots\alpha_{p-1}}dx_{\alpha_1}\dots dx_{\alpha_{p-1}},$$

where

$$C_{\alpha_1\dots\alpha_{p-1}}=\sum_{i=1}^{n}A_{\alpha_1\dots\alpha_{p-1},\,i}X_i\,.$$

By specializing this result so as to apply to the above canonical system and its I_2 we have

$$I_2=\int dt\int -dH.$$

Now, if this integral be extended to any E_2 for which $H=$ const., we have

$$I_2=0, \quad J_1=0.$$

It is to be noted that in this transformation of I_2 the form has been simplified *by means of the differential equations.* We can now prove the following:

THEOREM VI. *The relative invariant J_1 represents twice the action in a dynamical system, that is, $\int T dt$.*

We have $H=T-U$ where T is independent of the q_i and homogeneous of the second degree in the p_i, and U is a function of the q_i only. Then we have

$$J_1=\int\Sigma p_i dq_i=\int\Sigma p_i\frac{\partial H}{\partial p_i}\,dt=\int 2T dt\,.$$

UNIVERSITY OF ILLINOIS, *September,* 1916.

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[*] Goursat, *Journal de Mathématiques pures et appliquées,* 7 série, Tome I (1915).